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DIFFERENTIATION EXPLAINED

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

В. Г. Болтянский

ЧТО ТАКОЕ ДИФФЕРЕНЦИРОВАНИЕ?

«Физматгиз» Москва

LITTLE MATHEMATICS LIBRARY

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DIFFERENTIATION EXPLAINED

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AUTHOR'S PREFACE

High school students, especially those interested in mathematics, physics and engineering, often ask, 'What is "higher" mathematics?' Sometimes they discuss this and similar questions at mathematics clubs at schools.

In this book I have tried to explain, in a way a high school pupil would understand, certain concepts of higher mathematics*) such as the derivative, differential equation, the number e, and natural logarithm (pupils are more apt to be aware of and interested in the latter two concepts). Wherever possible, I have tried to illustrate the concepts with problems taken from physics. In addition, I have tried to show that the concepts of "higher mathematics" are mathematical reflections of actual processes, that mathematics and life are connected, not separated, and that mathematics is a growing, not an unchanging, completed science.

Not all proofs and arguments are presented with complete mathematical rigour. Some arguments are presented for illustration. This method seems to me more appropriate for a general book.

The book can be used by mathematics and physics clubs at school. Part of the material is taken from lectures the author gave at the request of the advisers of school mathematics clubs at the Moscow State University.

I would like to take this opportunity to express my sincere gratitude to A. I. Markushevich and A. Z. Ryvkin for their valuable advice and comments on the manuscript.

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*) The reader may become acquainted with certain concepts of higher mathematics by reading other books of this series: A I. Markushevich *Ploshchadi i logariphmy* (Squares and Logarithms) (No. 9) and I. P. Natanson *Summirovanie beskonechno malykh velichin* (The Summation of Infinitesimals) (No 12)

The Problem of a Free Falling Body

Statement of the Problem

We shall first estimate the *velocity of a body that falls to the ground from a certain height.*

Elementary physics tells us that a body falling freely in a vacuum obtains after t seconds from the start of the process the velocity

$$v = v_0 + gt \tag{1}$$

where v_0 is the initial velocity and g the gravitational acceleration.

In certain cases formula (1) remains approximately valid for a body falling in the atmosphere (as distinct from a vacuum), in other cases it can lead to gross errors. For example, it is applicable to a body falling a short distance. When a body falls from a great height, the expression of velocity greatly differs from formula (1). In 1945 V. G. Romanyuk in a free-fall jump fell 12 000 m before he opened his parachute. In a vacuum a body which fell from such a height (without initial velocity) would at ground level have attained a speed of around 500 m/s. In fact, it follows from the formula $s = \frac{gt^2}{2}$ that falling time equals (in a vacuum)

$$t = \sqrt{\frac{2s}{g}} \approx \sqrt{\frac{2 \times 12\,000 \text{ m}}{9.8 \text{ m/s}^2}} \approx 49.5 \text{ s}$$

and from (1) we obtain the speed

$$v = gt \approx 9.8 \text{ m/s}^2 \times 49.5 \text{ s} \approx 485 \text{ m/s}$$

(We could have used the formula $v^2 = 2gs$ right away.) At the same time it has been established that the speed of the parachutist in the course of a free-fall jump reaches 50-60 m/s and does not exceed this value. Thus, in this case formula (1) leads us to an erroneous conclusion.

Here is another example: the parachute is designed so that a man descending with it approaches the ground with a speed of

approximately 6.5 m/s no matter from what altitude the descent is being made.

Clearly, formula (1) is in this case of no value.

This points to the conclusion that the speed of a body falling in the atmosphere *reaches a constant value* in the course of time. In other words, some time after the start of the fall the motion of the body becomes uniform and its acceleration becomes zero. This means that the resultant force (the sum of all forces) acting on the body is zero.

It is easily seen why formula (1) is of no use for calculating the velocity of a body falling in the atmosphere. The formula is deduced on the assumption that the body moves under the action of only one force, namely, the *force of gravity*

$$P = mg \tag{2}$$

On the other hand, we have established that for a body falling in the atmosphere the resultant force (some time after the motion has begun) becomes zero, i. e. the force of gravity *P is compensated* by some other force which was not taken into account when formula (1) was deduced. This compensating force is the *air resistance*, or *drag*. It is the drag which prevents the parachutist from attaining excessive speed – it "supports" him, so to speak.

How can drag be taken into account? Suppose there is no wind. For a stationary body the drag is zero. The faster the body moves the more difficult it becomes for it to penetrate the air, i. e. the drag increases. This fact may be easily demonstrated on a still day if you increase your speed: start walking, then running, cycling, etc. We shall suppose that the magnitude of this force *is proportional to the velocity*, i. e. that it equals bv, where b is the proportionality factor and v the speed of motion. This assumption is well supported by experiment for low speeds*) not exceeding 1-2 m/s. The factor b depends on the dimensions and on the shape of the body. For instance, the speeds being equal, the drag of a ball is approximately 20 times that of a "cigar-shaped" body of equal cross section (Fig. 1).

Confining ourselves to these brief remarks, in future we shall assume that the drag (for which we shall adopt the designation S) is

$$S = -bv \tag{3}$$

the minus sign shows that the drag has a direction opposite to that of velocity.

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*) Note that for speeds above 1-2 m/s the drag exceeds the value bv. It is sometimes assumed that the drag is proportional to the *square* of velocity

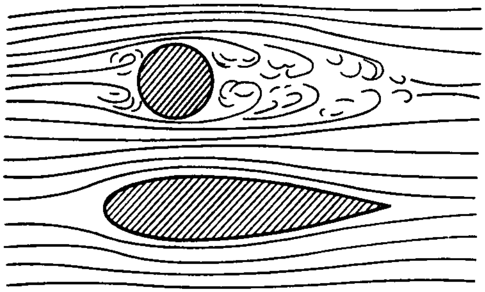

Fig 1

All said, we shall assume that a body thrown downwards with an initial velocity is acted upon by only two forces: the force of gravity P and the drag S. Making use of Newton's Second Law we may write:

$$ma = P + S \tag{4}$$

where m is the mass of the body and a its acceleration. It is more convenient to point the positive direction vertically *downwards* than upwards since the velocity of a falling body is directed downwards, and, if this convention is adopted, it will be positive. The force of gravity which is also directed downwards will also be positive. On the other hand, the drag will have the direction opposite to the velocity, i. e. upwards, and will for this reason be negative. Hence, substituting the values (2) and (3) of P and S into formula (4) we obtain

$$ma = mg - bv$$

or

$$a = -\frac{b}{m}\left(v - \frac{mg}{b}\right) \tag{5}$$

It is natural to assume the acceleration to be positive if it is directed downwards and negative if it is directed upwards.

Equation (5) relates the acceleration to the speed of motion which we do not know yet. It is this equation which we shall have to use to determine how the velocity of the moving body will change in time.

Qualitative Solution of the Problem

As a result of the above discussion we have obtained *equation* (5) for the velocity of a falling body. Now it is up to us to *solve* the equation. For this reason the following arguments are of a purely mathematical nature, although to help the reader see the point we shall continue to treat the matter of the velocity of a falling body.

Equation (5) connects two unknown quantities: velocity and acceleration. Assigning an arbitrary value to the acceleration we shall, using equation (5), obtain a definite value for the velocity. Therefore, our first impression is that *one* equation (5) is not enough to find *two* values v and a.

However, this impression is erroneous. The acceleration of a body is completely determined by the manner in which its velocity changes with time. Therefore equation (5) contains not completely arbitrary values, but *related* values v and a. This makes it possible to solve equation (5). Studying the relation between the velocity and the acceleration, we shall later on arrive at the concept of the derivative.

We shall prove two properties of velocity which follow from equation (5); these properties will make the process perfectly clear (in the assumptions made above). In due course we shall also obtain an accurate formula for the velocity.

PROPERTY 1 *If initially the velocity v_0 of a falling body is less than $\frac{mg}{b}$, then in the course of motion the relation $v \leqslant \frac{mg}{b}$ always holds. If; on the other hand, $v_0 > \frac{mg}{b}$, then $v \geqslant \frac{mg}{b}$ will always be true.*

Suppose this is not so. For instance, let $v_0 < \frac{mg}{b}$, and at a certain instant t_1 (i. e. t_1 seconds after the body started falling) let the velocity exceed $\frac{mg}{b}$. It follows that at some intermediate moment (perhaps, on numerous occasions) the velocity was *equal* to $\frac{mg}{b}$. Let t_0 be the *last* instant (in the first t_1 seconds) when the velocity was equal to $\frac{mg}{b}$, so that the inequality $v > \frac{mg}{b}$ held for the interval $t_0 - t_1$. It follows from formula (5) that during all this time the acceleration a was *negative*. However, this contradicts the statement that within the corresponding interval of time the velocity has *exceeded* $\frac{mg}{b}$. The resulting contradiction proves that the velocity cannot exceed the value of $\frac{mg}{b}$.

The case $v_0 > \frac{mg}{b}$ should be considered in the same way.

PROPERTY 2 *If $v_0 < \frac{mg}{b}$, the velocity of a falling body increases with time, all the time approaching the value $\frac{mg}{b}$; if, on the other*

hand, $v_0 > \frac{mg}{b}$, *the velocity decreases steadily also approaching the value* $\frac{mg}{b}$.

Evidently, if, for instance, $v_0 > \frac{mg}{b}$, the relation $v \geqslant \frac{mg}{b}$ will, as we know from Property 1, hold for the motion as a whole. It follows from formula (5) that the acceleration will be negative and therefore the velocity will steadily decrease.

Let us prove that in the course of time the difference $v - \frac{mg}{b}$ will become less than any predetermined small quantity h (this may be, for instance, made equal to 0.001 m/s). Therefore, consider the instant

$$t^* = \frac{\left(v_0 - \frac{mg}{b}\right)m}{hb}$$

During the time from the start of the motion up to the instant t^* the velocity of the falling body has decreased from the value v_0 to a value not less than $\frac{mg}{b}$, i. e. it has decreased no less than by an amount

$$v_0 - \frac{mg}{b} = \frac{hb}{m}t^*$$

It follows that at some intermediate instant the acceleration was not greater than $\frac{hb}{m}$, for should the acceleration have been greater during the whole interval of time the velocity would have decreased more than by the amount $\frac{hb}{m}t^*$.

So let us assume that at the instant t' $|a| < \frac{hb}{m}$

In accordance with (5) it follows that

$$\left|v - \frac{mg}{b}\right| = \frac{m}{b}|a| \leqslant \frac{m}{b}\frac{hb}{m} = h$$

i. e. at the instant t' the difference between the velocity and the value $\frac{mg}{b}$ is less than h. This will hold for all future instants as well, for the velocity continues to decrease always remaining greater than $\frac{mg}{b}$.

It should be noted that we have thus proved a somewhat more precise statement than Property 2, namely, that *not later than after the time*

$$t^* = \left| v_0 - \frac{mg}{b} \right| \frac{m}{hb} \text{ s} \tag{6}$$

after the fall has started the difference between the velocity and the value $\frac{mg}{b}$ *will be less than h.*

Properties 1 and 2 provide us in a certain sense with a solution of the problem we have set ourselves. Although we have not yet obtained a precise formula for the velocity, we have, nevertheless, found out *qualitative* laws for the variation of the velocity, i. e. we know now *how* it is going to change with time.

For instance, let us observe the motion of the parachutist. If he opens his parachute the moment he jumps, his velocity will increase from zero, but shall never exceed the value $\frac{mg}{b}$. The value mg (the weight of the jumper with the parachute) is known, and b depends on the parachute's diameter. This makes it possible to *calculate* the dimensions of the parachute so that the maximum possible speed of descent equal to $\frac{mg}{b}$ guarantees a safe landing for the parachutist. On the other hand, in case of a free-fall jump (with the parachute tucked away) the coefficient for the drag will have a new value, which we shall denote by b' and which is smaller than in case of a descent with an open parachute For this reason the maximum speed $\frac{mg}{b'}$ is greater in the latter case. It follows that before the parachute is opened in the course of a free-fall jump the parachutist's speed will be greater than $\frac{mg}{b}$ and, according to Property 1, after the parachute is opened it will decrease approaching $\frac{mg}{b}$, at the same time remaining greater than $\frac{mg}{b}$. Thus in this case, too, some time after the parachute has been opened the jumper is able to make a safe landing.

Let us take a numerical illustration.

Example 1. Let a parachute be designed so that a jumper's speed with the parachute open approaches the limiting value of 6 m/s,

i. e. $\frac{mg}{b} = 6$ m/s. The jumper opens his parachute while falling at a speed of 50 m/s. How much time will lapse before his speed becomes less than 10 m/s, i. e. differs from the limiting value $\frac{mg}{b} = 6$ m/s by an amount less than $h = 4$ m/s?

Solution. From the equation $\frac{mg}{b} = 6$ m/s we obtain:

$$\frac{m}{b} = \frac{mg}{b}\frac{1}{g} \approx \frac{6\ \text{m/s}}{10\ \text{m/s}^2} = 0.6\ \text{s}$$

Next using formula (6) we conclude that the speed of descent will differ from the limiting value $\frac{mg}{b} = 6$ m/s by $h = 4$ m/s not later than after a time $\left(v_0 - \frac{mg}{b}\right)\frac{m}{b}\frac{1}{h}$ seconds, i. e. with the assumptions made, after

$$(50\ \text{m/s} - 6\ \text{m/s}) \times 0.6\ \text{s} \times \frac{1}{4\ \text{m/s}} = 6.6\ \text{s}$$

Formula for the Velocity of a Falling Body. The Number *e*

Properties 1 and 2 demonstrate how the velocity of a falling body changes with time. In this section we intend to obtain an exact formula for the velocity of a falling body. A certain number enters into the expression for the velocity whose value within the fifth decimal place is 2.71828... . Frequently used in "higher" mathematics, this number is denoted by the letter e (in the same way as another frequently used number 3.14159..., which expresses the ratio of the circumference to the diameter, is denoted by the letter π). Later we shall see why this number $e = 2.71828...$ enters into the formula for the velocity and how it should be accurately calculated. For the present, we shall write the formula for the velocity of a falling body without deduction and shall consider some examples illustrating its application.

Let v_0 be the initial velocity of a falling body and v_t the velocity of the body at the instant t (i. e. t seconds after the body has

started falling). *Then we have*

$$v_t = \frac{mg}{b} + \left(v_0 - \frac{mg}{b}\right) e^{-\frac{b}{m}t} \tag{7}$$

This is the exact solution of equation (5). The proof of formula (7) will be given below. Let us consider some examples.

Example 2. Let us demonstrate that formula (7) immediately leads to the qualitative laws for the variation of the velocity (Properties 1 and 2) obtained above.

The number $e^{-\frac{b}{m}}$, which is obtained by raising the number e to a negative power, is, in effect, positive and less than unity, i. e. $0 < e^{-\frac{b}{m}} < 1$. As t increases, $e^{-\frac{b}{m}t} = \left(e^{-\frac{b}{m}}\right)^t$ decreases (and may be made for sufficiently large t's as small as desired). Therefore it follows from formula (7) that, for instance, for $v_0 > \frac{mg}{b}$ the velocity v_t always exceeds the value $\frac{mg}{b}$ $\left(\text{for } v_0 - \frac{mg}{b} > 0\right)$ and that it decreases with time approaching $\frac{mg}{b}$.

Example 3. Let us calculate, using formula (7), the velocity of the parachutist 6.6 seconds after he opens his parachute in a free-fall jump. We shall use the same numerical values as in Example 1 (p. 12), i. e. $\frac{mg}{b} = 6$ m/s, $v_0 = 50$ m/s. (We have seen that this velocity must be less than 10 m/s.)

Solution. We have

$$\frac{b}{m} = g : \frac{mg}{b} \approx 10 \text{ m/s}^2 : 6 \text{ m/s} = \frac{5}{3}\,\frac{1}{\text{s}}$$

Next, using the tables of logarithms (the common logarithm of the number e is approximately 0.4343), we easily obtain

$$\log_{10}\left(e^{-\frac{b}{m}t}\right) = -\frac{b}{m}t\log_{10} e \approx -\frac{5}{3} \times 6.6 \times 0.4343 =$$
$$= -4.7773 = -5 + 0.2227 = \bar{5}.2227$$

whence

$$e^{-\frac{b}{m}t} \approx 0.0000167$$

Substituting this value into formula (7), we obtain

$$v_{6\,6\,s} \approx 6 \text{ m/s} + (50 \text{ m/s} - 6 \text{ m/s}) \times 0.0000167 \approx 6.000735 \text{ m/s}$$

Similarly, with the aid of formula (7), it may be easily calculated that the parachutist speed will be 10 m/s (the conditions being the same) $t = \dfrac{3 \log_{10} 11}{5 \log_{10} e}$ s ≈ 1.44 s after the parachute opens.*)

Thus, *when the parachute in a free-fall jump is opened, the speed of descent decreases in* 1-2 s *from* 50-60 m/s *almost to the normal speed of* 6-7 m/s *under an open parachute.* Here the parachutist slows down considerably, i. e. undergoes the great force of the upward jerk of the parachute, caused chiefly by air resistance. One who has watched free-falls at aviation holidays, for example, may have seen how a person rapidly plunging to the ground, suddenly, the moment the parachute opens, slows down sharply; it may even seem that for an instance he dangles in midair.

Example 4. Let a parachutist's rate of descent in a free-fall jump approach the limiting value of $\dfrac{mg}{b} = 50$ m/s. We shall set the speed at the start at zero. What will the error be if instead of formula (7) we use formula (1), applicable for a body falling in a vacuum?

Solution We have

$$-\frac{b}{m} = -\frac{g}{\frac{mg}{b}} \approx -\frac{10 \text{ m/s}^2}{50 \text{ m/s}} = -0.2\ \frac{1}{\text{s}}$$

Thus, in accordance with formula (7), the parachutist's rate of descent will be

$$v_t = 50\,(1 - e^{-0.2t})$$

We obtain the following value of the speed of a body falling in a vacuum from formula (1).

$$v_t = gt \approx 10t$$

Thus the ratio of the velocities becomes

$$\frac{50\,(1 - e^{-0.2t})}{gt} \approx \frac{5}{t}(1 - e^{-0.2t})$$

---

*) The speed of descent will in fact approach the limiting value of 6 m/s still more rapidly since expression (3) for the drag is really valid only for small velocities. For greater speeds the drag's dependence on speed is *greater* than bv

Putting $t = 1$ s, we obtain (after some easy calculations with the aid of the logarithmic tables) for the above ratio the value ≈ 0.91, and for $t = 2$ s – the value ≈ 0.82. We see that *already the first few seconds after the body has started falling, its speed (because of the drag) differs noticeably from the value gt.*

Now let us prove formula (7). Therefore, let us first elucidate the relation between the velocity and the acceleration. If v_t is the velocity of the body's motion at the instant t, and v_{t+h} its velocity h seconds after this instant (i. e. at the instant $t + h$), the ratio $\frac{v_{t+h} - v_t}{h}$ is termed the *average acceleration* of the body in the interval h and denoted by a_{av}:

$$a_{av} = \frac{v_{t+h} - v_t}{h}$$

If h is very small (say, 0.01 s, or smaller, depending on the nature of the body's motion) the acceleration changes little during such a small interval, so that a_{av} differs little from the value a_t at the instant t. The difference between a_t and a_{av} is the smaller the less is h. In other words, should we put h progressively smaller (say, 0.1 s, 0.01 s, 0.001 s, etc.), leaving t constant, a_{av} would change approaching a_t closer with every step. In mathematics this fact is written down as follows:

$$a_t = \lim_{h \to 0} a_{av} = \lim_{h \to 0} \frac{v_{t+h} - v_t}{h}$$

The sign lim means the *limit* of the expression which follows it (i. e. the expression a_{av}); the designation $h \to 0$ below indicates that we have in mind the limit a_{av} when h tends to zero.

Thus we have obtained the relation expressing the dependence of the acceleration on the velocity. Let us now prove three additional properties of the velocity of motion we have been discussing. These properties will help us to prove formula (7).

PROPERTY 3 *If the velocity and the acceleration of a moving body satisfy relation* (5), *this means that further variation of the velocity is uniquely determined by the initial velocity* v_0.

Suppose the contrary is true. Let two bodies T and T^* with equal values of m and b move in such a manner that their velocities and accelerations satisfy relation (5). Suppose at the instant $t = 0$ these bodies had equal initial velocities v_0, but t_1 seconds later their velocities became different, say the velocity of the first body v_1 became greater than the velocity of the second

body v_1^*. To be definite, suppose that $v_0 > \frac{mg}{b}$ (should the opposite hold, the proof would be quite similar). Let t_0 be the last instant (during the first t_1 seconds) when the velocities of both bodies were equal. In that case in the period $t_1 - t_0$ the velocity of the first body v was all the time greater than the velocity of the second body v^*, i. e. $v > v^*$. Hence it follows that

$$v - \frac{mg}{b} > v^* - \frac{mg}{b}$$

both quantities $v - \frac{mg}{b}$ and $v^* - \frac{mg}{b}$ being positive since $v_0 > \frac{mg}{b}$ (see Property 1). In accordance with formula (5) we conclude from the inequalities

$$v - \frac{mg}{b} > v^* - \frac{mg}{b} > 0$$

that the accelerations of both bodies a and a^* are negative, the magnitude of a being greater than that of a^*. This, however, means that the velocities of the bodies T and T^* have decreased during the period $t_1 - t_0$, the velocity of the body T having decreased *more* than that of the body T^*, i. e. at the instant t_1 the velocity v should be *less* than v^* (because at the instant t_0 these velocities coincided). We, however, made the opposite assumption. The resulting contradiction proves that Property 3 is correct.

PROPERTY 4. *If two identical*) bodies T and T^* start falling simultaneously with initial velocities v_0 and v_0^*, their velocities v_t and v_t^* at any instant t will satisfy the relation*

$$\frac{v_t^* - \frac{mg}{b}}{v_t - \frac{mg}{b}} = \frac{v_0^* - \frac{mg}{b}}{v_0 - \frac{mg}{b}} \tag{8}$$

To prove this consider an imaginary body $\tilde{T}$ which moves so that at the instant t its velocity is equal to

$$\tilde{v}_t = qv_t + (1 - q)\frac{mg}{b}$$

---

*) In the sense that their respective m's and b's are equal.

where

$$q = \frac{v_0^* - \frac{mg}{b}}{v_0 - \frac{mg}{b}}$$

We shall demonstrate that the velocity and the acceleration of this imaginary body will satisfy relation (5). Find the average acceleration $\tilde{a}_{av}$ of this imaginary body inside the interval t, $t+h$. We have

$$\tilde{a}_{av} = \frac{\tilde{v}_{t+h} - \tilde{v}_t}{h} = \frac{\left[qv_{t+h} + (1-q)\frac{mg}{b}\right] - \left[qv_t + (1-q)\frac{mg}{b}\right]}{h} =$$

$$= q\,\frac{v_{t+h} - v_t}{h} = qa_{av}$$

where a_{av} is the average acceleration of the body T inside the same interval. Should we take progressively smaller values of h in the relation $\tilde{a}_{av} = qa_{av}$, $\tilde{a}_{av}$ would approach the acceleration $\tilde{a}_t$ of the imaginary body at the instant t, and a_{av} the acceleration a_t of the body T at the same instant. In this way we shall obtain $\tilde{a}_t = qa_t$ (for an arbitrary instant t) and with the aid of equation (5)

$$\tilde{a} = qa = -q\frac{b}{m}\left(v - \frac{mg}{b}\right) =$$

$$= -\frac{b}{m}\left[qv + (1-q)\frac{mg}{b} - \frac{mg}{b}\right] = -\frac{b}{m}\left(\tilde{v} - \frac{mg}{b}\right)$$

i. e. relation (5) is satisfied for the imaginary motion we have been discussing above.

Next the initial velocity of the imaginary body $\tilde{T}$ is equal to

$$\tilde{v}_0 = qv_0 + (1-q)\frac{mg}{b} = q\left(v_0 - \frac{mg}{b}\right) + \frac{mg}{b} =$$

$$= \frac{v_0^* - \frac{mg}{b}}{v_0 - \frac{mg}{b}}\left(v_0 - \frac{mg}{b}\right) + \frac{mg}{b} = v_0^*$$

Thus both bodies $\tilde{T}$ and T^* have identical initial velocities and both move so that their velocities and accelerations satisfy equation (5). In accordance with Property 3 it follows that their velocities $\tilde{v}_t$ and v_t^* coincide at every instant t, i. e.

$$v_t^* = qv_t + (1-q)\frac{mg}{b}$$

Hence we obtain

$$\frac{v_t^* - \frac{mg}{b}}{v_t - \frac{mg}{b}} = \frac{\left[qv_t + (1-q)\frac{mg}{b}\right] - \frac{mg}{b}}{v_t - \frac{mg}{b}} = \frac{q\left(v_t - \frac{mg}{b}\right)}{v_t - \frac{mg}{b}} = q =$$

$$= \frac{v_0^* - \frac{mg}{b}}{v_0 - \frac{mg}{b}}$$

thereby proving Property 4.

PROPERTY 5 *For any instants t and τ the relation holds*

$$\frac{v_{t+\tau} - \frac{mg}{b}}{v_0 - \frac{mg}{b}} = \frac{v_t - \frac{mg}{b}}{v_0 - \frac{mg}{b}} \frac{v_\tau - \frac{mg}{b}}{v_0 - \frac{mg}{b}} \tag{9}$$

where v_0, v_τ, v_t, $v_{t+\tau}$ are the velocities of the falling body T at the instants 0, τ, t, $t+\tau$.

Indeed, start observing how the body T is falling from the instant τ onwards. In t seconds (i. e. $t+\tau$ seconds after the motion has started) the velocity will be $v_{t+\tau}$. This means that if at the instant $t=0$ we in addition to the body T drop another body T^* whose initial velocity v_0^* equals v_τ, the velocity of the latter body v_t^* at the instant t will be equal to $v_{t+\tau}$, i. e. $v_t^* = v_{t+\tau}$. Hence we obtain from (8)

$$\frac{v_{t+\tau} - \frac{mg}{b}}{v_t - \frac{mg}{b}} = \frac{v_\tau - \frac{mg}{b}}{v_0 - \frac{mg}{b}}$$

or otherwise

$$\left(v_{t+\tau} - \frac{mg}{b}\right)\left(v_0 - \frac{mg}{b}\right) = \left(v_t - \frac{mg}{b}\right)\left(v_\tau - \frac{mg}{b}\right)$$

Dividing both parts of the equation thus obtained by $\left(v_0 - \frac{mg}{b}\right)^2$, we obtain the required relation (9).

After deriving formula (9) let us start to calculate the exact value of the velocity v_t. In order to avoid lengthy formulas let us, as a temporary measure, introduce the notation

$$u_t = \frac{v_t - \frac{mg}{b}}{v_0 - \frac{mg}{b}}$$

thus simplifying formula (9):

$$u_{t+\tau} = u_t u_\tau \qquad (10)$$

For $\tau = t$ formula (10) yields

$$u_{2t} = u_t^2$$

Similarly, putting $\tau = 2t$, we obtain from (10)

$$u_{3t} = u_t u_{2t} = u_t u_t^2 = u_t^3$$

and for $\tau = 3t$ the result will be

$$u_{4t} = u_t u_{3t} = u_t u_t^3 = u_t^4$$

etc. Continuing in the same way, we shall find that for any positive integer n the relation holds

$$u_{nt} = u_t^n \qquad (11)$$

Setting in this equation $t = \frac{p}{n}$ s and extracting the root of the n-th power, we obtain

$$u_p^{\frac{1}{n}} = u_{\frac{p}{n}}$$

Next setting in equation (11) $t = 1$ s and substituting p for n, we obtain

$$u_p = u_1^p$$

It follows from the last two equations that

$$u_{\frac{p}{n}} = u_1^{\frac{p}{n}}$$

Thus, for any positive rational number t (i. e. the number of the form $\frac{p}{n}$, where p and n are positive integers) it holds

$$u_t = u_1^t$$

or returning to the previous notation

$$\frac{v_t - \frac{mg}{b}}{v_0 - \frac{mg}{b}} = \left(\frac{v_1 - \frac{mg}{b}}{v_0 - \frac{mg}{b}}\right)^t \tag{12}$$

Here v_1 is the velocity of the falling body at the instant $t = 1$ s.

Inasmuch as relation (12) is valid for rational t's it is valid for all t's.

Let us take, for example, the instant $t = \sqrt{2}$ s $= 1.414...$ s. The numbers 1.4, 1.41, 1.414, etc. are rational and for this reason relation (12) holds for these values of t:

$$\frac{v_{1.4} - \frac{mg}{b}}{v_0 - \frac{mg}{b}} = \left(\frac{v_1 - \frac{mg}{b}}{v_0 - \frac{mg}{b}}\right)^{1.4},$$

$$\frac{v_{1.41} - \frac{mg}{b}}{v_0 - \frac{mg}{b}} = \left(\frac{v_1 - \frac{mg}{b}}{v_0 - \frac{mg}{b}}\right)^{1.41}, \ldots \tag{13}$$

Should we set t equal to rational values each of which is progressively a more exact approximation of the number $\sqrt{2}$ the left-hand side of equation (13) would approach the limit $\dfrac{v_{\sqrt{2}} - \frac{mg}{b}}{v_0 - \frac{mg}{b}}$ and the right-hand side the limit $\left(\dfrac{v_1 - \frac{mg}{b}}{v_0 - \frac{mg}{b}}\right)^{\sqrt{2}}$.*)

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*) See, for example, the concept of an irrational power in the course of algebra by A. P. Kiselev, part II (p. 97 in the 1960 Russian edition).

Thus we obtain in the limit

$$\frac{v_{\sqrt{2}} - \frac{mg}{b}}{v_0 - \frac{mg}{b}} = \left(\frac{v_1 - \frac{mg}{b}}{v_0 - \frac{mg}{b}}\right)^{\sqrt{2}}$$

Similar treatment is, of course, applicable not only to $\sqrt{2}$, but to any irrational value of t. Hence, relation (12) is valid for any t.

Introducing the notation

$$\frac{v_1 - \frac{mg}{b}}{v_0 - \frac{mg}{b}} = c$$

we obtain from (12)

$$\frac{v_t - \frac{mg}{b}}{v_0 - \frac{mg}{b}} = c^t$$

whence

$$v_t = \frac{mg}{b} + \left(v_0 - \frac{mg}{b}\right)c^t \tag{14}$$

Formula (14) obtained above for the velocity of a falling body is not the final one since we do not know the value of the number c which enters this formula. To calculate the number c, let us find from formula (14) the acceleration of the body at the initial instant of its motion. The average acceleration for the first h seconds has according to (14) the following value

$$a_{\text{av}} = \frac{v_h - v_0}{h} = \frac{\frac{mg}{b} + \left(v_0 - \frac{mg}{b}\right)c^h - v_0}{h} = \left(v_0 - \frac{mg}{b}\right)\frac{c^h - 1}{h}$$

As h tends to zero, this expression gives us the acceleration a_0 at the initial instant:

$$a_0 = \lim_{h \to 0} a_{\text{av}} = \lim_{h \to 0}\left(v_0 - \frac{mg}{b}\right)\frac{c^h - 1}{h} \tag{15}$$

Denoting the expression $c^h - 1$ by x, we obtain

$$c^h - 1 = x, \qquad c^h = 1 + x, \qquad h = \log_c(1 + x)$$

Thus instead of the expression $\left(v_0 - \frac{mg}{b}\right)\frac{c^h - 1}{h}$, which enters (15) under the limit sign, we obtain the expression

$$\left(v_0 - \frac{mg}{b}\right)\frac{x}{\log_c(1+x)} = \frac{v_0 - \frac{mg}{b}}{\frac{1}{x}\log_c(1+x)} = \frac{v_0 - \frac{mg}{b}}{\log_c(1+x)^{\frac{1}{x}}}$$

Note that as h tends to zero the limit of the number c^h is unity and that the number $x = c^h - 1$ approaches zero. Hence we may write:

$$a_0 = \lim_{x \to 0} \frac{v_0 - \frac{mg}{b}}{\log_c(1+x)^{\frac{1}{x}}} \tag{16}$$

The limit of the expression $(1 + x)^{\frac{1}{x}}$ *as* x *tends to zero is termed the number* e. We shall leave *without proof* that this limit exists, i. e. that the expression $(1 + x)^{\frac{1}{x}}$ does in fact approach some value as $x \to 0$. The proof (by the way, quite elementary) of the existence of this limit may be found in the first chapters of any textbook on higher mathematics.*)

We shall confine ourselves to calculating the values $(1 + x)^{\frac{1}{x}}$ for $x = 0.1$, $x = 0.01$, $x = 0.001$, and $x = 0.0001$. The results of such calculations are presented below (these calculations may be performed with the aid of logarithmic tables, preferably seven-place ones; one may also use Newton's binomial formula):

$$(1 + 0.1)^{\frac{1}{0.1}} = 1.1^{10} \approx 2.59374$$

$$(1 + 0.01)^{\frac{1}{0.01}} = 1.01^{100} \approx 2.70481$$

$$(1 + 0.001)^{\frac{1}{0.001}} = 1.001^{1000} \approx 2.71692$$

$$(1 + 0.0001)^{\frac{1}{0.0001}} = 1.0001^{10000} \approx 2.71814$$

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*) See, for instance, V I. Smirnov, *Kurs vysshei matematiki* (A Course in Higher Mathematics), V. 1

These calculations clearly demonstrate that as $x \to 0$ the limit of the expression $(1+x)^{\frac{1}{x}}$ is $e = 2.71\ldots$. We now obtain from (16)

$$a_0 = \frac{v_0 - \frac{mg}{b}}{\log_c e}$$

On the other hand, it follows from (5) that

$$a_0 = -\frac{b}{m}\left(v_0 - \frac{mg}{b}\right)$$

Equating the expressions obtained for a_0, we obtain

$$\frac{v_0 - \frac{mg}{b}}{\log_c e} = -\frac{b}{m}\left(v_0 - \frac{mg}{b}\right)$$

whence

$$\log_c e = -\frac{m}{b}, \qquad e = c^{-\frac{m}{b}}, \qquad c = e^{-\frac{b}{m}}$$

Finally substituting $e^{-\frac{b}{m}}$ for c into formula (14) we obtain formula (7) thus concluding the proof.

Differentiation

The Concept of the Derivative

Equation (5) as we have just seen may be solved quite accurately. This equation relates the quantity v (the velocity of a falling body) with the quantity a which shows *how rapidly the quantity v changes* (acceleration is "the rate of change of speed").

When we talk of the rate of change of some quantity, we assume that the quantity we are dealing with is not a constant characterized by a number, but a *variable* quantity, i. e. a quantity whose value changes with time. Here are some examples of quantities that change with time (or depend on time): the velocity and the acceleration of non-uniform motion, the magnitude of alternating current, etc.

Let y be a quantity whose value changes with time. For the sake of convenience we shall denote by y_t the value this quantity assumes t seconds after the process under observation has started. The difference $y_{t+h} - y_t$ shows how much the variable y has changed in h seconds (in the interval t, $t + h$ seconds after the process has started). The ratio

$$\frac{y_{t+h} - y_t}{h} \tag{17}$$

for its part shows how much y changes on the average every second (within this interval of time), i.e. this ratio represents the *average rate of change of the variable y*. Putting h progressively smaller, we shall obtain the values of the average rate over progressively shorter intervals starting from the instant t. As a limiting case (when h tends to zero) relation (17) yields the *rate of change of the variable y* at the instant t. We know already that this rate of change is designated in mathematics by

$$\lim_{h \to 0} \frac{y_{t+h} - y_t}{h} \tag{18}$$

Expression (18) is termed the *derivative* of the quantity y with respect to time t, and as we have seen it represents the rate of change of the variable y. We may deal with a variable which changes not with time, but depends on some other quantity. For instance, the area of a circle depends on its radius. Denoting the area of a circle of radius R by S_R, we obtain

$$S_R = \pi R^2 \tag{19}$$

Should we investigate the dependence of the circle's area on its radius we would arrive at the ratio $\frac{S_{R+h} - S_R}{h}$, which expresses the mean rate of change of the area with the change of radius. The limit of this ratio (as $h \to 0$) is the derivative of S with respect to R.

The concept of the derivative is one of the basic concepts of higher mathematics. In cases when the variable y changes in accordance with variations of a quantity x (or when, as may also be said, y is a function of x) the derivative of y with respect to x is denoted by y', or more often, by one of the symbols $\frac{dy}{dx}$, $\frac{d}{dx}y$:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y_{x+h} - y_x}{h}$$

here the letter d should not be cancelled out since it is not a multiplicand but *denotes the process of finding the derivative*, or, as is also said, *the differentiation.*

Let us by way of example calculate the derivative $\frac{dS}{dR}$ of function (19):

$$\frac{dS}{dR} = \lim_{h \to 0} \frac{S_{R+h} - S_R}{h} = \lim_{h \to 0} \frac{\pi (R + h)^2 - \pi R^2}{h} =$$

$$= \lim_{h \to 0} (2\pi R + \pi h) = 2\pi R$$

i. e. the derivative of the circle's area with respect to its radius equals its circumference.

For the second example let us find the derivative $\frac{ds}{dt}$ of the distance with respect to time. Denote the distance covered by

a body up to the instant t (i. e. t seconds after it has started moving) by s_t. Then the ratio $\frac{s_{t+h} - s_t}{h}$ will represent the average speed during the interval t, $t + h$, the limit of this ratio for $h \to 0$ being the value of the speed at the instant t:

$$v_t = \lim_{h \to 0} \frac{s_{t+h} - s_t}{h} = \frac{ds}{dt}$$

In the same way one may calculate the derivative $\frac{dv}{dt}$. The ratio

$$\frac{v_{t+h} - v_t}{h}$$

is the average acceleration during the interval t, $t + h$, the limit of this ratio being the value of the acceleration at the instant t (compare with what was said on p. 16):

$$a_t = \lim_{h \to 0} \frac{v_{t+h} - v_t}{h} = \frac{dv}{dt}$$

The relations that have just been proved

$$v = \frac{ds}{dt} \tag{20}$$

and

$$a = \frac{dv}{dt} \tag{21}$$

play an extremely important part in mechanics.

The Differential Equation

Let us again turn to equation (5). According to (21) this equation may be rewritten in the form

$$\frac{dv}{dt} = -\frac{b}{m}\left(v - \frac{mg}{b}\right) \tag{22}$$

Now it is evident that this is an equation with one unknown quantity v. However, this is not an *algebraic equation*, but an equation relating the quantity v to its derivative. Such an equation is termed a *differential equation*. Comparing differential equation (22) with its solution (7) and denoting $\frac{b}{m}$ by k and $\frac{mg}{b}$ by c, we may make the following statement.

THEOREM. The solution of the differential equation

$$\frac{dv}{dt} = -k(v - c) \tag{23}$$

is represented by the expression

$$v = c + (v_0 - c)e^{-kt} \tag{24}$$

where v_0 is the initial value of v (i. e. its value for $t = 0$).

In future we shall be able, using this theorem, to calculate some other physical phenomena, as well.

Two Problems Leading to Differential Equations

(a) Switching on an Electric Current. Consider an electric circuit consisting of a coil and a battery (Fig. 2). The electrical

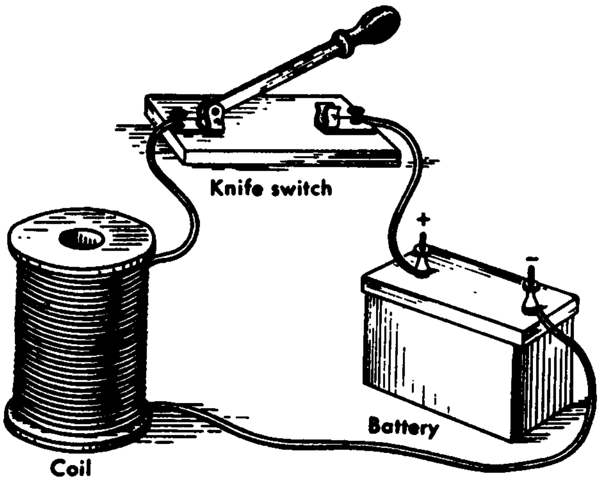

Fig. 2

properties of a coil are rather complex, but for some cases they may be described with great accuracy with the aid of two quantities: its *resistance* and its *inductance*. The coil is in fact usually represented by two sections connected in series: by the resistance and the inductance (Fig. 3). The voltage drop across the resistance is proportional to the *current* i passing through the coil (Ohm's Law)

$$V = Ri$$

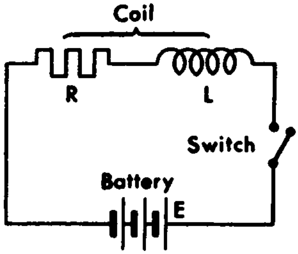

Fig 3

The proportionality factor R is termed the *resistance* of the coil. The voltage drop across the inductance is proportional to the *rate of current variation.* Denoting the current variation rate by w (it is measured, for example, in amperes per second) and the proportionality factor by L, we obtain for the voltage drop the expression

$$V = Lw$$

The quantity L is termed the *inductance* of the coil. The voltage drop across the coil is the sum of voltage drops across the resistance and the inductance, i. e. it is expressed by the formula

$$V = Lw + Ri \tag{25}$$

Formula (25) is well supported by experiment (if the frequency of the current passing through the coil is not too high). This is the formula we are going to use. Denote the electromotive force (emf) of the battery by E. Equating the emf of the battery to the voltage drop across the coil, we obtain on the basis of Kirchhoff's Second Law the equation (we neglect the internal resistance of the battery and the resistance of the conductors):

$$E = Lw + Ri$$

or

$$w = -\frac{R}{L}\left(i - \frac{E}{R}\right) \tag{26}$$

The solution of this equation may be easily obtained from the theorem formulated on p. 28. Indeed, denoting the current at the instant t by i_t, we may contend that the quantity

$$w_{av} = \frac{i_{t+h} - i_t}{h}$$

is the average rate of change in the current during the interval t, $t + h$. When $h \to 0$ we obtain the rate of the current change

at the instant t

$$w = \lim_{h \to 0} \frac{i_{t+h} - i_t}{h} = \frac{di}{dt}$$

Hence the quantity w is the derivative of the current i, and equation (26) may be rewritten in the form

$$\frac{di}{dt} = -\frac{R}{L}\left(i - \frac{E}{R}\right)$$

The only difference between this equation and equation (23) is that the variable we are seeking is denoted by i and not by v which is, of course, inessential. The constants k and c contained in equation (23) in this case assume the following values

$$k = \frac{R}{L}, \qquad c = \frac{E}{R}$$

Hence the solution of this differential equation will be of the form (see (24)):

$$i_t = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-\frac{R}{L}t}$$

Should the current i_0 the instant the battery was switched on ($t = 0$) be zero the formula would assume a simpler form

$$i_t = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

It follows from this formula that the current, zero at the instant of switching on, grows continuously approaching the value $\frac{E}{R}$, i. e. the value of the current which would pass through the coil if it had the same resistance R and no inductance at all.

(b) Radioactive Decay. Suppose we have a piece of rock which contains some radioactive material. The atoms of the radioactive material decay and are transformed into another chemical substance – the *product of decay*. Hence, in the course of time the amount of the radioactive material contained in the piece of rock decreases. Let us introduce the concept of the *decay rate*. Suppose at some instant t the amount of the radioactive material contained in the rock had been m_t grammes and h years later it decreased (because of the decay) to m_{t+h} g. The expression

$$\frac{m_{t+h} - m_t}{h}$$

shows by what amount, in grammes per year, the mass of the radioactive material has on the average decreased (during the respective period of time). It would be natural to term this expression the *average* (*mean*) *decay rate* for the period. The limit to which this expression tends as $h \to 0$ is the decay rate at the instant t. Denote it by u

$$u = \lim_{h \to 0} \frac{m_{t+h} - m_t}{h} = \frac{dm}{dt}$$

Note that the decay rate is *negative* since the mass of radioactive material decreases with time.

What does the decay rate depend on? When the amount of radioactive material contained in the rock is small, it may be accepted that the *decay rate is proportional to the amount of radioactive material contained in the piece at a certain instant*, i. e.

$$u = -km$$

where m is the mass of radioactive material and k a positive constant (proportionality factor).

The arguments in favour of approximate validity of this law are simple if one assumes that the decay of some atoms of radioactive material does not affect the state of the others. In these conditions it may be assumed that the amount of radioactive material decaying per unit time out of every gramme is always approximately the same, say k grammes, no matter how much radioactive material still remains in the rock. In this case the amount of radioactive material decaying per unit time out of m g will be km g.

Is the assumption that the decay does not affect the state of the remaining radioactive atoms correct? One should bear in mind that the decay products may strike another atom, thereby causing it to decay; new particles will thus be generated – this may lead to the decay of further atoms, etc. Such a *chain reaction* (this, for example, is the basic process in the functioning of the atom bomb) would not fit with the assumption of the independent decay of the atoms. To make this chain impossible, the particles emitted in the course of the decay should be lost and should not (in most cases) strike other radioactive atoms. This will be the case if radioactive material makes up only a small percentage of the mass, the bulk of the rock being non-radioactive. Then the overwhelming majority of the particles emitted in the course of the decay will be lost in collisions with non-radioactive atoms of the rock, and a chain reaction will not be possible. Therefore, for small amounts of radioactive material contained in the rock the decay of the

individual atoms may, to a good approximation, be assumed to be independent.

We obtain accordingly the following differential equation for evaluating the mass of the remaining radioactive material

$$\frac{dm}{dt} = -km$$

which differs from equation (23) in that the unknown quantity is designated by the letter m instead of v, and that the constant c is in this particular case zero. Hence, according to (24) the solution will take the form

$$m_t = m_0 e^{-kt} \qquad (27)$$

where m_0 is the mass of the radioactive material at the initial instant (when we begin to take interest in the process of decay).

Example 5. How many years will it take for the amount of the radioactive material to decrease by half?

Solution. To obtain an answer to this problem, we should determine t from the equation

$$m_0 e^{-kt} = \frac{1}{2} m_0$$

Cancelling out m_0 and taking the logarithm, we find

$$t = \frac{1}{k} \log_e 2 \approx \frac{0.69}{k}$$

This period is termed *half-life* of the corresponding radioactive material. Note that this period does not depend on the *initial amount* of the radioactive material, but only on k, i. e. on the *nature* of the material. For instance, half-life of radium is 1590 years, that of uranium 238 – about 4.5 thousand million years.

Example 6. Formula (27) permits us to draw certain conclusions concerning the *age of the Earth.*

Suppose a piece of rock extracted from the depths of the Earth contains together with some inclusions m g of the radioactive material and p g of its decay product. Suppose in addition that every gramme of this radioactive material yields (upon complete decay) r g of the decay product. This means that p g of the decay product was produced from $\frac{p}{r}$ g of the radioactive material. Hence if we assume that the radioactive decay process started at some instant (i. e. up to that instant the material had not contained a single atom of the decay product), we must put the initial mass of the radioactive material equal to $m + \frac{p}{r}$. To calculate the time that has passed from that imaginary instant (since the beginning

of the decay) until today, we must in accordance with (27) solve the equation

$$m = \left(m + \frac{p}{r}\right) e^{-kt}$$

with respect to t, whence we obtain

$$t = \frac{1}{k} \log_e \left(1 + \frac{p}{rm}\right)$$

Such calculations performed for some minerals contained in the Earth yield approximately the same value for t – around 2×10^9 years. Thus *the conditions on the Earth that permitted a normal decay process persisted for several thousand million years.* It may be conjectured that some thousand million years ago the matter which now makes up the Earth existed under quite different conditions which facilitated the production of radioactive atoms from the simpler atoms and from other particles.

Napierian Logarithms

The formulas for solving the problems presented above contain the exponential function with the base e. When making calculations based on these formulas with the aid of logarithmic tables, one can exclude some operations if one uses *logarithms with the base e*. For instance, taking the logarithm in formula (27) with the base e and with the base 10, we obtain

$$\log_e m_t = -kt + \log_e m_0, \qquad \log_{10} m_t = -kt \log_{10} e + \log_{10} m_0$$

In the second instance an extra logarithm has to be taken and an extra multiplication performed. Moreover, problems result in formulas which contain logarithms with the base e as we have seen in Examples 5 and 6. The number e comes up in other mathematical problems as well, and it is very convenient to use logarithms with the base e, especially in theoretical questions. *Logarithms with the base e are termed Napierian, or natural, logarithms and denoted by the symbol* ln: the meaning of the expression $\ln x$ is the same as that of $\log_e x$. Common and natural logarithms are related by the expression

$$\log_{10} x = M \cdot \ln x$$

where

$$M = \log_{10} e \approx 0.4343$$

This relation is easily obtained if one takes the logarithm with the base 10 of the identity $e^{\ln x} = x$

Harmonic Oscillations

The Problem of Small Oscillations of a Pendulum

Suppose a thread of length l is fixed at point C with a body M suspended from its other end (a pendulum). The problem is to find out the nature of the motion of the body M. To solve this problem mathematically, we shall simplify it somewhat. First of all we shall assume that the thread from which the pendulum M is suspended is *inextensible* and *weightless*.

We shall examine the motion of the pendulum which takes place in a single vertical plane passing through the suspension point. Because the thread is inextensible we may be sure that the body M will move along a circumference of radius l with the centre at point C. The assumption about the negligible mass of the thread means that its weight is negligible as compared with that of the body M. This makes it possible for us to assume that external forces act only on the body M. We shall assume the pendulum M to be a heavy point (i. e. we shall attribute to it a mass m and neglect its dimensions). From all the forces that act on the body M we shall in addition to the tension of the thread take into account only the force of gravity. When considering this problem, we may also neglect the air resistance (for instance, we may presume the body M to be confined to a hermetic evacuated vessel). The problem of how much the motion of the pendulum in the atmosphere differs from its motion in a vacuum is treated in the note on p. 49.

Let the body M at a certain instant be at point A on the circumference along which it is moving. Denote the lowermost point of the circumference by Q, the length of arc QA by s, and the central angle $\angle QCA$ corresponding to it (Fig. 4) by α (in radians). Then

$$s = l\alpha \tag{28}$$

We shall assume the arc s and the angle α to be positive when point A is to the right of Q and negative in the opposite case.

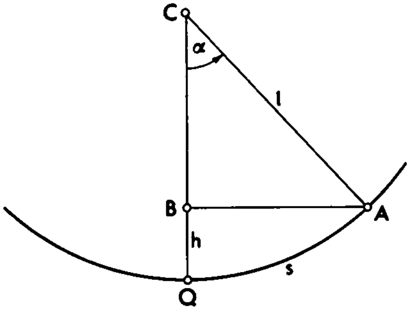

Fig 4

Now let us deduce the equation from which we are to learn the law of the pendulum's motion. The difference in altitude of points A and Q is

$$h = QB = CQ - CB = l - l\cos\alpha = l(1 - \cos\alpha) = l \cdot 2\sin^2\frac{\alpha}{2}$$

Assuming the potential energy of the pendulum to be zero at point Q, we find for the potential energy of the pendulum at point A the value

$$W^{(p)} = mgh = mg2l\sin^2\frac{\alpha}{2}$$

The kinetic energy of the pendulum is

$$W^{(k)} = \frac{mv^2}{2}$$

where v is the velocity of the body M.

Hence the total energy E of the pendulum (at point A) is expressed by the formula

$$E = \frac{mv^2}{2} + 2mgl\sin^2\frac{\alpha}{2} \tag{29}$$

The pendulum does not perform any work in its motion (we neglected friction and drag!), and for this reason its energy E remains all the time constant.

We shall somewhat simplify equation (29) by treating the problem of *small oscillations of the pendulum*, i. e. of such motion of the pendulum in the course of which the deflection angle from its equilibrium position Q remains small. Let us clarify what we mean by "small angles". The point is that an exact solution of

equation (29) cannot be written with the aid of any known operations. Therefore the question arises as to whether equation (29) may be replaced by a simpler one. Naturally this simplification should be such that the solution of the simplified equation would be a very accurate approximation to the solution of equation (29). Note that hereby we do not *in principle* introduce a new inaccuracy since relation (29) is in itself approximate*), so the applicability of an approximation of a definite sort depends only on the degree of accuracy we want to obtain.

The usual simplification to which equation (29) is subjected is the substitution of an angle φ for $\sin\varphi$. That such a substitution is permissible for *small* angles φ will be seen from Fig. 5, which

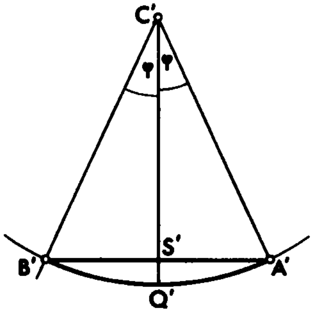

Fig 5

shows an arc $A'Q'B'$ of the circumference of radius $C'Q' = 1$; the angle φ is plotted on both sides of radius $C'Q'$. The length of section $A'B'$ is $2\sin\varphi$ (for $A'S'$ is the sine line), the length of arc $A'B'$ being 2φ (the angles are, of course, measured in radians!). It is clear from the figure that small angles φ differ little, this difference being the less the smaller is φ. For instance, it may readily be checked with the aid of trigonometric tables that for angles not exceeding 0.245 radian (i. e. $\approx 14°$) the ratio

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*) In deducing equation (29) we have made some simplifications: neglected the drag, the weight of the thread, the dimensions of the body M, etc.

Note that any physical law or any mathematical relation linking physical quantities (for example, relations (1), (2), (3), (4), (5), (25), and (29)) is an approximate one, for in every case there are "forces" which were not taken into account when the physical law or mathematical relation was formulated. This, however, does in no way belittle the enormous importance of physical laws. For instance, the accuracy of Newton's Second Law and of Ohm's Law is under ordinary conditions extremely great.

$\frac{\sin\varphi}{\varphi}$ differs from unity by less than 0.01; for angles less than 1° (0.017 radian) this difference drops to about 0.0005.

Accordingly, assuming the deflection angle of the pendulum to be small, we substitute $\frac{\alpha}{2}$ for $\sin\frac{\alpha}{2}$, i. e. we substitute a new equation for equation (29) which differs little from the former:

$$\frac{mv^2}{2} + 2mgl\left(\frac{\alpha}{2}\right)^2 = E$$

Taking into account (28), we may write this equation in the form

$$\frac{mv^2}{2} + \frac{mgs^2}{2l} = E$$

or

$$\frac{l}{g}v^2 + s^2 = \frac{2lE}{mg} \tag{30}$$

This equation contains two unknown variables: s and v (we assume that constants g, l, m, and E are known). Yet this equation (same as equation (5)) may be solved, for quantities s and v are not arbitrary but related by expression (20). It follows from (20) that equation (30) may be written in the form

$$\frac{l}{g}\left(\frac{ds}{dt}\right)^2 + s^2 = \frac{2lE}{mg} \tag{31}$$

so it is, in effect, an equation of *one* unknown. Let us solve this equation.

Let us choose a coordinate system lying in a plane with the value of s measured along the x- and the value of $\sqrt{\frac{l}{g}}\,v$ along the y-axes. At any instant t there is a definite value of the distance s and the velocity v of a body M, i. e. a definite point N in the plane (Fig. 6). Conversely, if we know where point N lies, we can

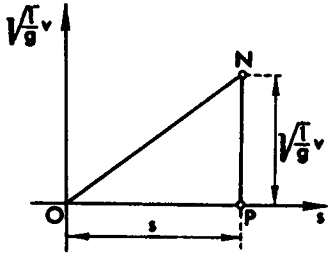

Fig 6

find its coordinates s and $\sqrt{\frac{l}{g}}\,v$, i. e. we can learn the position of the pendulum and its velocity. Thus at any instant t the pendulum M is symbolized by some point N. The length of section ON is easily calculated with the aid of the Pythagorean theorem:

$$ON = \sqrt{PN^2 + OP^2} = \sqrt{\frac{l}{g}v^2 + s^2}$$

i. e. (in accordance with relation (30))

$$ON = \sqrt{\frac{2lE}{mg}}$$

As the pendulum swings, the values s and v will change, i. e. point N will move across the plane which contains the coordinate system. But the distance from point N to the origin of coordinates always remains the same, i. e. equal to $\sqrt{\frac{2lE}{mg}}$. Hence point N moves along a circumference of the radius

$$R = \sqrt{\frac{2lE}{mg}} \tag{32}$$

This circumference is termed *phase circumference*.

Let us find the velocity with which point N moves along the circumference. This velocity is directed along a tangent to the circumference. Let it be represented, for example, by a vector NA (Fig. 7). Let us decompose the vector NA into a horizontal and a vertical components. Then a horizontal component NB will represent the velocity of motion of point P along the x-axis. The distance from point P to O being s, the velocity of motion of point P is $\frac{ds}{dt} = v$, i. e. $NB = v$.

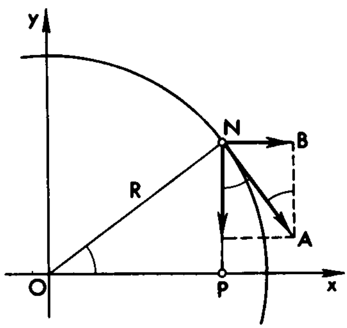

Fig 7

Now since triangles ONP and NAB are similar, we obtain

$$PN:ON = NB:NA \quad \text{or} \quad \sqrt{\frac{l}{g}}\,v:R = v:NA$$

From the latter proportion we obtain

$$NA = R\sqrt{\frac{g}{l}}$$

Such is the velocity with which point N moves along the circumference.

Denote the initial deflection and the initial velocity of a pendulum by s_0 and v_0, respectively, and the corresponding point on the phase circumference by N_0. Then the radius of the phase circumference will be

$$R = \sqrt{\frac{l}{g}v_0^2 + s_0^2} \tag{33}$$

(see (30) and (32)). An angle $\varphi_0 = \angle XON_0$ will be determined from the relation

$$\tan\varphi_0 = \frac{\sqrt{\frac{l}{g}}\,v_0}{s_0} \tag{34}$$

(Fig. 8). Next, t seconds after the motion of the pendulum has

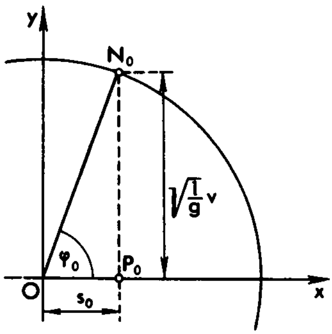

Fig 8

started, point N moving at a velocity $R\sqrt{\frac{g}{l}}$ will travel a distance $N_0N = R\sqrt{\frac{g}{l}}\,t$ along the phase circumference, and an angle

$\angle N_0ON$ will accordingly be $\sqrt{\frac{g}{l}}t$. As a result (Fig. 9)

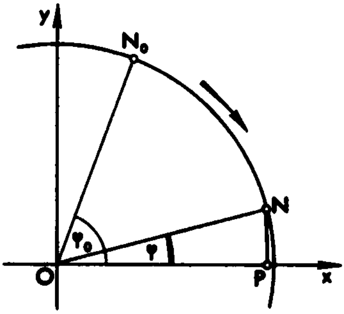

Fig. 9

$$\varphi = \angle XON = \angle XON_0 - \angle N_0ON = \varphi_0 - \sqrt{\frac{g}{l}}t$$

Hence we obtain

$$OP = R\cos\varphi = R\cos\left(\varphi_0 - \sqrt{\frac{g}{l}}t\right) = R\cos\left(\sqrt{\frac{g}{l}}t - \varphi_0\right)$$

$$PN = R\sin\varphi = R\sin\left(\varphi_0 - \sqrt{\frac{g}{l}}t\right) = -R\sin\left(\sqrt{\frac{g}{l}}t - \varphi_0\right)$$

Remembering finally that $OP = s$ and $PN = \sqrt{\frac{l}{g}}v$, we obtain

$$s = R\cos\left(\sqrt{\frac{g}{l}}t - \varphi_0\right), \quad v = -\sqrt{\frac{g}{l}}R\sin\left(\sqrt{\frac{g}{l}}t - \varphi_0\right) \quad (35)$$

These formulas express the deflection and the velocity of the pendulum t seconds after it has started to swing, i. e. they represent a complete solution of problem of pendulum's motion (with the simplifications we have made). Let us consider some examples.

Example 7. The pendulum is initially deflected to the right to a distance s_0 and dropped with a zero initial velocity. Find its deflection and velocity at the instant t.

Solution. In this case $R = s_0$, $\varphi_0 = 0$, and from formulas (35) we obtain

$$s = s_0\cos\sqrt{\frac{g}{l}}t, \qquad v = -\sqrt{\frac{g}{l}}s_0\sin\sqrt{\frac{g}{l}}t$$

Example 8. The pendulum was initially in the equilibrium position Q and was thrown out of this position by a thrust which imparted to it the initial velocity v_0 directed to the right (i. e.

a positive initial velocity). Find the deflection and the velocity at the instant t.

Solution. From formulas (33) and (34) we obtain in this case that $R = \sqrt{\frac{l}{g}}\, v_0$, $\varphi_0 = \frac{\pi}{2}$, and from formulas (35):

$$s = \sqrt{\frac{l}{g}}\, v_0 \cos\left(\sqrt{\frac{g}{l}}\, t - \frac{\pi}{2}\right) = \sqrt{\frac{l}{g}}\, v_0 \sin\sqrt{\frac{g}{l}}\, t$$

$$v = -v_0 \sin\left(\sqrt{\frac{g}{l}}\, t - \frac{\pi}{2}\right) = v_0 \cos\sqrt{\frac{g}{l}}\, t$$

Example 9. Let us find the derivatives of functions $\sin \omega t$ and $\cos \omega t$.

Since v is the time derivative of s, comparing the values of s and v in Example 8, we conclude that

$$\frac{d}{dt}\left(\sqrt{\frac{l}{g}}\, v_0 \sin\sqrt{\frac{g}{l}}\, t\right) = v_0 \cos\sqrt{\frac{g}{l}}\, t$$

Similarly, from Example 7 we obtain

$$\frac{d}{dt}\left(s_0 \cos\sqrt{\frac{g}{l}}\, t\right) = -\sqrt{\frac{g}{l}}\, s_0 \sin\sqrt{\frac{g}{l}}\, t$$

Setting, for instance, $v_0 = \sqrt{\frac{g}{l}}$, $s_0 = 1$ and denoting the quantity $\sqrt{\frac{g}{l}}$ by ω, we obtain from these formulas:

$$\frac{d}{dt}\sin \omega t = \omega \cos \omega t, \qquad \frac{d}{dt}\cos \omega t = -\omega \sin \omega t \tag{36}$$

Example 10. Since cosine and sine are periodic functions after some time T, termed the *period of oscillations*, the pendulum again returns to the initial position and again repeats the same motion. Let us find the period of oscillations of the pendulum.

Solution. The values of sine and cosine remain unchanged when their arguments change by 2π. Therefore the period of oscillations of the pendulum is the period T during which the expression in parentheses following the sine and cosine symbols in equations (35) increases by 2π. In other words, the difference between the values of the expression $\sqrt{\frac{g}{l}}\, t - \varphi_0$ at the instants t and $t + T$

must be 2π

$$\sqrt{\frac{g}{l}}(t+T)-\varphi_0=\left(\sqrt{\frac{g}{l}}t-\varphi_0\right)+2\pi$$

From this relation T is easily obtained

$$T=2\pi\sqrt{\frac{l}{g}} \tag{37}$$

In this way the motion is periodically repeated every T seconds. The pendulum *oscillates periodically*. During each period (i. e. every T seconds) the pendulum, as may be seen from (35), once assumes the extreme right-hand position (cosine becomes equal to $+1$) and once the extreme left-hand position (cosine becomes equal to -1). At these instants of maximum deflection the velocity of the pendulum turns zero (see (35)). When cosine turns ± 1, the sine of the same argument turns zero. The maximum velocity of the pendulum is attained (sine turns ± 1) when it passes through point Q (cosine turns zero).

The Differential Equation of Harmonic Oscillations

We have deduced formulas describing the motion of the pendulum from equation (30) or, which is the same, differential equation (31). There is another differential equation which, too, describes the motion of the pendulum we have been discussing above. Its deduction is very simple.

Suppose a body M at some instant is at point A of the circumference along which it moves. Let us decompose the force of gravity (which we assume to be equal to mg and to be directed vertically downwards), using the parallelogram rule, into two components: one tangent to the circumference at point A and the other perpendicular to this tangent. The component which is perpendicular to the tangent tends to stretch the thread and is compensated by the tension of the thread (for the thread is presumed to be inextensible). A force F acts along the tangent in the direction of point Q and is equal in value, as may readily be seen, to $mg \sin\alpha$ (Fig. 10), i. e. the force F is negative for positive α's, and vice versa. Thus

$$F=-mg\sin\alpha$$

If both mutually compensating forces – the tension of the thread and the component of the force of gravity perpendicular to the tangent – are ignored, there remains only the force F acting on the

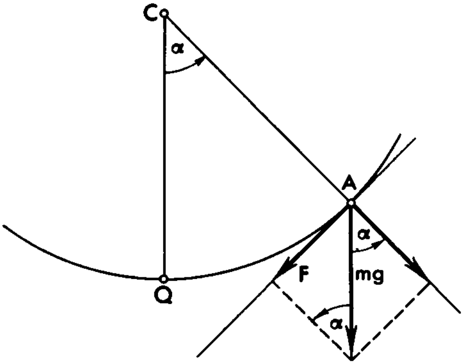

Fig 10

body M (we neglected the air resistance), and we may therefore write on the basis of Newton's Second Law

$$ma = -mg \sin \alpha$$

or

$$a = -g \sin \alpha$$

Recalling that we are now dealing only with *small oscillations* of the pendulum, and that this makes it possible to approximate $\sin \alpha$ by α, we may write this equation in the form

$$a = -g\alpha$$

or according to (28)

$$a + \frac{g}{l} s = 0 \qquad (38)$$

This is just the required equation. Let us show how it may be written in the form of a *differential equation*. It follows from relations $a = \frac{dv}{dt}$ and $v = \frac{ds}{dt}$ that if we take a derivative of a distance s once and then take the second derivative of the first one obtained (i. e. of the velocity), we shall obtain the acceleration. In other words, the acceleration is the *second derivative* of the distance s (with respect to time t). This is written down as follows

$$a = \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

or

$$a = \frac{d^2 s}{dt^2} \qquad (39)$$

The symbol $\frac{d^2s}{dt^2}$ (the second derivative of s with respect to t) is not treated as an algebraic expression, but as a single symbol; we are not allowed to perform any operations with its components (for instance, to cancel anything out of the "fraction"). It follows from (39) that equation (38) may be written in the form of a differential equation as follows

$$\frac{d^2s}{dt^2} + \frac{g}{l}s = 0 \tag{40}$$

Note that we *are already able* to solve this equation. Indeed, it describes the law of variation of the quantity s, i. e. the law of the oscillations of the pendulum, and this we have already investigated. Therefore we may state at once that the solution of equation (40) is contained in the first formula of (35). We may express this in more detail as follows.

The solution for differential equation (40) *is*

$$s = R\cos\left(\sqrt{\frac{g}{l}}\,t - \varphi_0\right)$$

where R and φ_0 are given by formulas (33) *and* (34). Note that to find the quantities R and φ_0 we must know the initial deflection s_0 and the initial velocity v_0, i. e. the values of s and $\frac{ds}{dt}$ taken at the initial instant.

Denote a quantity $\sqrt{\frac{g}{l}}$ by ω. Then we may express the statement made above in the following form.

THEOREM. *The differential equation*

$$\frac{d^2s}{dt^2} + \omega^2 s = 0 \tag{41}$$

has the solution

$$s = R\cos(\omega t - \varphi_0) \tag{42}$$

where R and φ_0 depend on the initial values of s and $\frac{ds}{dt}$.

Equation (41) is known as the *equation of harmonic oscillations.* Any quantity described by this equation is said to be oscillating harmonically; this means that the respective quantity changes with time in accordance with the law (42). The quantity ω which enters

differential equation (41) and its solution (42) is termed *angular oscillation frequency*, and $T = \frac{2\pi}{\omega}$ *oscillation period*. When the quantity s oscillates harmonically, any value it assumes is repeated every T seconds (see Example 10).

Let us compare differential equations (23) and (41). Equation (23) contains only the first derivative, therefore it is termed a *first-order differential equation*. Equation (41) is a *second-order differential equation* because it contains the second derivative. Note that only the initial value of the quantity v itself had to be known to solve the first-order equation (23). To solve the second-order equation (41), on the other hand, we must know not only the initial value of the quantity s itself, but of its derivative $\frac{ds}{dt}$ as well. In short, to solve a first-order equation we must know the initial value of only one quantity; to solve a second-order equation, of two.

Note that we have obtained the solution of equation (40) from physical considerations: both equations (31) and (40) describe the same physical phenomenon, and for this reason they should have identical solutions expressing the law of pendulum's oscillations. Of course, such argumentation is guesswork and not a rigorous mathematical proof. It may be proved by purely mathematical means that equations (31) and (40) are equivalent, i. e. that their solutions are the same: after differentiating both sides of equation (31) we obtain equation (40). Conversely, one may obtain equation (31) from equation (40), but then one will have to use an operation reverse to that of differentiation. Such an operation (termed *integration*) forms together with differentiation the basis of higher mathematics as a whole. A more detailed discussion is beyond the scope of this booklet.

Still, by applying formulas (36), the reader will easily find the second derivative of function (42) and make sure for himself that the function satisfies equation (41).

Let us consider two examples from the field of physics which lead to the equation of harmonic oscillations.

The Oscillatory Circuit

Let us consider an oscillatory circuit, i. e. a closed electric circuit consisting of a coil and a capacitor. The coil possesses an inductance and a resistance (see p. 29). The complete circuit may be represented in the form of a circuit diagram (Fig. 11). Let us denote by q the electric charge which goes over from one plate of the capacitor

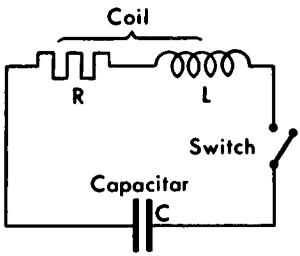

Fig 11

to another and by i the current in the circuit. (We presume that initially the capacitor had a charge q_0, and that a current i_0 was in the coil, and are interested in subsequent changes in these quantities.) In that case the voltage across the capacitor will be $\frac{q}{C}$, where C is its capacitance; the voltage drop across the coil will be equal to $Lw + Ri$, where R is the resistance and L the inductance (see (25)). According to Kirchhoff's Second Law the sum of the voltage drops in the circuit is zero, i. e.

$$Lw + Ri + \frac{q}{C} = 0 \tag{43}$$

A quantity i is the derivative of q with respect to t. Indeed, if the values of the charge q were q_t and q_{t+h} at the instants t and $t + h$, respectively, this means that during that interval a charge passed through the cross section of the circuit's wire (anywhere in the circuit) equal to $q_{t+h} - q_t$. Therefore the average current over the interval t, $t + h$ was

$$i_{\text{av}} = \frac{q_{t+h} - q_t}{h}$$

Taking the limit, we obtain

$$i = \lim_{h \to 0} \frac{q_{t+h} - q_t}{h} = \frac{dq}{dt}$$

It follows from relations $i = \frac{dq}{dt}$ and $w = \frac{di}{dt}$ that w is the derivative of $i = \frac{dq}{dt}$, i. e. that w is the second derivative of q:

$$w = \frac{d^2q}{dt^2}$$

Hence equation (43) may be rewritten in the form:

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=0 \tag{44}$$

The differential equation we have just obtained is more complex than equation (41), since in addition to the unknown function q and its second derivative $\frac{d^2q}{dt^2}$ it contains also the first derivative $\frac{dq}{dt}$. We shall not, however, try to solve equation (44) (see the note on p. 49), but shall only discuss the case when the resistance of the coil R is negligible (compared to the quantities L and C), neglecting on this account the term $R\frac{dq}{dt}$ in equation (44). Then the equation will assume the form:

$$L\frac{d^2q}{dt^2}+\frac{q}{C}=0$$

or

$$\frac{d^2q}{dt^2}+\frac{1}{LC}q=0 \tag{45}$$

Equation (45) is, evidently, an equation of harmonic oscillations (see (41)) with the oscillation frequency of the circuit

$$\omega=\frac{1}{\sqrt{LC}}$$

and the period of oscillations expressed by the formula

$$T=2\pi\sqrt{LC}$$

The solution of equation (45) is of the form (see (42)):

$$q=R\cos\left(\frac{t}{\sqrt{LC}}-\varphi_0\right)$$

where R and φ_0 depend on the initial values, i. e. on q_0 and i_0.

Oscillations Produced by the Action of the Elastic Force of a Spring

Suppose a weight of mass m is suspended from a spring. The spring will be somewhat elongated under the action of the force of gravity (the elongation will stop as soon as the spring

tension compensates the force of gravity); in this position the weight and the spring can remain stationary (be in a state of equilibrium). Should we shift the weight from its equilibrium position pulling it downwards, the spring tension would exceed the force of gravity and the resultant force would be directed upwards. Should we, on the contrary, shift the weight to a point above the equilibrium position, the resultant force would be directed downwards. Thus, this resultant force "tries" to return the weight to its equilibrium position.

For the sake of simplicity we shall content ourselves with only the vertical motion of the weight: up and down. Denote the position of equilibrium by O, the position of the weight at some instant by A, and a distance OA by s, assuming the positive direction along the vertical to be downward from point O, i. e. we will take s to be positive if the weight (point A) is below point O, and negative if it is above point O. Denote the resultant force of the spring's gravity and tension by F, and the air resistance by S. We shall assume that no other forces except F and S act on the weight. In accordance with Newton's Second Law we may write:

$$ma = F + S$$

where a is the weight's acceleration. The force F which tends to return the weight to its equilibrium position is directly proportional to a deviation s, i. e. it is numerically equal to the quantity ks, where k is a proportionality factor. This assumption is well supported by experiment (for not very great deviations from the equilibrium position). The quantity k is termed the *stiffness of the spring*. For positive s (point A is below O) the force F is directed upwards, i. e. it is negative, for negative s the force F is positive. In other words, the sign of the force F is opposite to that of the deviation s, i. e.

$$F = -ks$$

We shall assume for the force S the same value as above (see (3)), i. e.

$$S = -bv$$

In this way we obtain the following equation of motion of the weight

$$ma = -ks - bv$$

or

$$ma + bv + ks = 0 \tag{46}$$

Since $v = \frac{ds}{dt}$ and $a = \frac{d^2s}{dt^2}$, this equation may be written in the form

$$m\frac{d^2s}{dt^2} + b\frac{ds}{dt} + ks = 0 \tag{47}$$

Differential equation (47) is analogous to differential equation (44), which we obtained while solving the problem of the oscillatory circuit. We shall not solve equation (47) (see the note below), but consider only the case when the air resistance may be neglected (i. e. when a quantity b is very small compared to quantities m and k). In that case equation (47) takes the form

$$\frac{d^2s}{dt^2} + \frac{k}{m}s = 0 \tag{48}$$

Relation (48) is an equation of harmonic oscillations with the frequency

$$\omega = \sqrt{\frac{k}{m}}$$

and the period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The solution of equation (48) is according to (42) of the form:

$$s = R\cos\left(\sqrt{\frac{k}{m}}t - \varphi_0\right) \tag{49}$$

where R and φ_0 depend on the initial conditions, i. e. on s_0 and v_0.

Note. In order to obtain the equation of harmonic oscillations we neglected the forces of friction and air resistance when studying the oscillations of a pendulum and a weight suspended from the spring, and the resistance of the coil when studying the oscillatory circuit. Physically it means that no energy is spent in the process of the oscillations, mathematically this leads to the omission of the term containing the first derivative. As a result we have obtained harmonic oscillations, i. e. oscillations that remain constant from cycle to cycle, or *sustained oscillations.*

What difference would it make if while solving the above problems we took into account the force of air resistance and

the voltage drop across the resistance? For instance, what is the difference between the solutions of equation (44) and those of equation (45)? Mathematical calculation (which we shall not present here) shows that when R is not too great, equation (44), too, describes an oscillatory process. The oscillations described by equation (44), however, become weaker with time; for this reason they are termed *damped oscillations*. The physical explanation of this difference is that the energy of oscillations constantly decreases being transformed into heat because current passing through the resistance R generates heat. In the same way the oscillations of the pendulum, too, become gradually weaker because due to friction and air resistance the pendulum's energy is gradually spent on heating the pendulum itself and the surrounding atmosphere. However, when the resistance is small, the difference between the damped and the sustained (harmonic) oscillations during a short interval of time (for example, for several periods) is not great. The damping makes itself felt only after a substantial time. Should we, for example, suspend a heavy weight from a string and deflect it slightly from its equilibrium position, the decrease in the amplitude of the oscillations after 10-15 periods would be quite negligible and unnoticeable to the eye. We would be able to observe it only several minutes after the oscillations have started.

Here for the purpose of comparison is the exact solution of equation (47) (without deduction). We shall presume that the value of the factor b in the expression for the force of air resistance is not too great (namely, $b < 2\sqrt{mk}$). Then the solution of equation (47) takes the form

$$s = Re^{-\frac{b}{2m}t} \cos\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\, t - \varphi_0\right) \tag{50}$$

where R and φ_0 are determined by the initial conditions. It may be seen from this formula that s decreases infinitely with time (multiplicand $e^{-\frac{b}{2m}t}$ becomes less and less as time t increases). Figure 12*a*, *b* shows the graphs of functions (50) for various values of a factor $\frac{b}{2m}$. The smaller is $\frac{b}{2m}$ the slower is the damping of the oscillations. Compare these graphs with those of harmonic oscillations (49) shown in Fig. 12*c* $\left(\text{formula (50) coincides with formula (49) for } \frac{b}{2m} = 0\right)$.

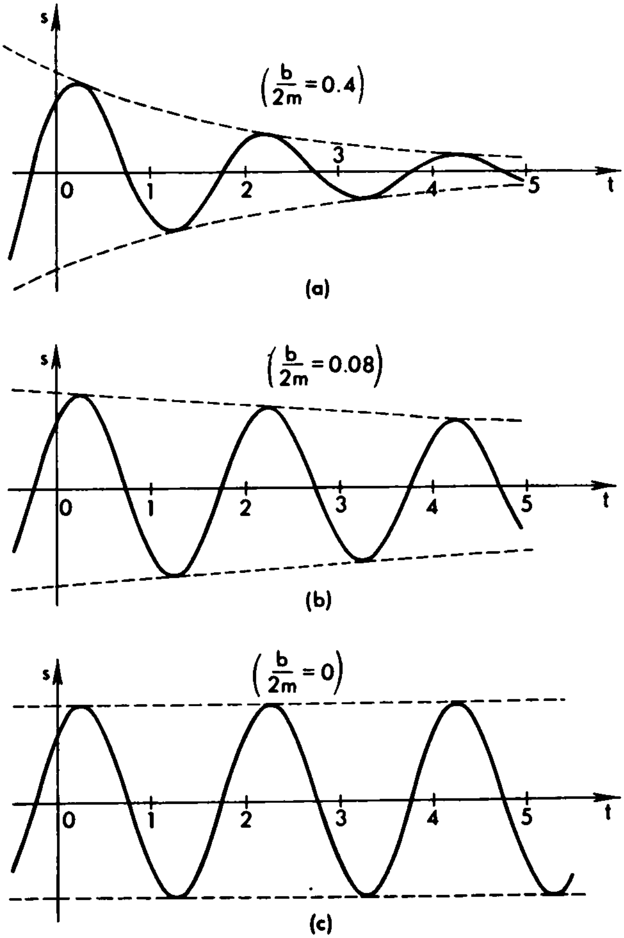

Fig 12

Note in addition that for large values of the factor b (for $b > 2\sqrt{mk}$) formula (50) is replaced by another formula. In this case the weight will pass through the equilibrium position not more than once and after that will slowly approach this position all the time remaining on the same side of it (below or above).

Some Other Applications of the Concept of the Derivative

Maximum and Minimum Values

Let us consider some variable quantity y whose value depends on some other quantity x. When you say that y depends on x, or that y is a *function* of x, you mean that there is a quite definite value of y to correspond to every value x assumes. For instance, the area of a circle is a function of its radius, i. e. the area of the circle depends on its radius. Sine, cosine, tangent, etc. depend on the value of an angle, i. e. they are functions of the angle. These functions are termed *trigonometric functions.*

So let us suppose that y is a function of x. Let us pose the following problem: find such a value of x that y would be at its *maximum.* Before proceeding with the solution, let us introduce the concept of the *domain of definition of a function.* We shall discuss this concept by using some examples.

For the first example let us take the following function. Suppose V is the volume of a kilogramme of water at normal atmospheric pressure and at a temperature $t°$ (Celsius). Here V depends on t, i. e. V is a function of the quantity t. Evidently, this function is defined only for t's lying inside the interval 0-100°. Indeed, at normal atmospheric pressure water cannot have a temperature $t < 0°$ (water will turn into ice) or $t > 100°$ (water will turn into steam). Hence, the function V is defined only for such values of t that satisfy the inequalities $t \geqslant 0$ and $t \leqslant 100$. Usually the two inequalities are written together: $0 \leqslant t \leqslant 100$. Thus the function V is defined only for

$$0 \leqslant t \leqslant 100$$

In other words, the *domain of definition* of the function V is made up of numbers which satisfy the condition $0 \leqslant t \leqslant 100$. Such a domain of definition is termed a *number segment* because, when numbers are represented on the number axis, all the numbers which satisfy the condition $0 \leqslant t \leqslant 100$ fill a complete segment of the number axis. The numbers 0 and 100 are termed the

terminals or *end points* of the number section $0 \leqslant t \leqslant 100$. All the other points of the section are termed its *internal values* or *internal points. It is peculiar to every internal value t_0 that its number segment contains both points greater and smaller than t_0.* The terminals of the segment do not possess that property.

For the second example let us consider the current i passing through the electric circuit (shown on the diagram in Fig. 3) t seconds after the circuit has been closed. In this case i is a function of time t. The dependence of i on t is shown by the formula presented on p. 30. For what values of t is the function i defined? Evidently, up to the instant the circuit was closed, i. e. for $t < 0$, there had been no current passing through it, therefore it is reasonable to consider the current i only for $t \geqslant 0$. Hence, the domain of definition of the function i will be made up of all the numbers that satisfy the condition $t \geqslant 0$. Such a domain of definition (it may be termed a *number semi-axis*) has only one end point $t = 0$, all the other points are internal.

Finally, for the third example let us consider a function $y = \sin x$. It is defined for any x, i. e. the domain of definition of this function is the entire number axis. This region has no end points.

There are functions whose domains of definition are very intricate. We, however, shall consider only such functions which have for their domain of definition the number segment, number axis or number semi-axis.

Let us now turn again to the problem of finding the maximum value of a function. Can a function attain its maximum value at an end point of its domain of definition? Of course, it can. Consider, by way of example, the function V discussed above, which defines the volume of a kilogramme of water at normal atmospheric pressure and at a temperature t°C. Since the volume of water increases upon heating (from 4°C upwards), it is clear that the function V will have its maximum value at $t = 100°$, i. e. at the end point of the domain of definition.

In many cases differentiation enables the position of the function's maximum to be determined quickly. Namely, the following proposition holds.

Let y be a function of the variable x. If this function attains its maximum value at an internal point $x = a$ of its domain of definition, the derivative $\frac{dy}{dx}$ at this point becomes zero).*

---

*) Subject to the condition that the derivative exists. There are such functions in mathematics that do not have a derivative

Let us prove this proposition. We shall denote the value of y that corresponds to the value of x (chosen from the domain of definition of the function) by y_x. We have presumed that the value y_a the function y assumes at $x = a$ is its maximum value, i. e.

$$y_a \geqslant y_x \tag{51}$$

for any x (chosen from the domain of definition of the function).

The derivative $\frac{dy}{dx}$ for $x = a$ is determined by the relation

$$\left.\frac{dy}{dx}\right|_{\text{for } x=a} = \lim_{h \to 0} \frac{y_{a+h} - y_a}{h} \tag{52}$$

Let us prove that this derivative is zero.

We shall start making h approach zero, attributing various *positive* values to it. As long as the numerator $y_{a+h} - y_a$ of the fraction under the limit sign satisfies the inequality $y_{a+h} - y_a \leqslant 0$ (see (51)), and $h > 0$, the fraction under the limit sign will itself be non-positive (i. e. it may be zero, or negative). But in that case the limit of this fraction, too, cannot be positive, i. e. *derivative* (52) *cannot be a positive number.*

Next let us make h approach zero, attributing various *negative* values to it. In this case, as before, $y_{a+h} - y_a \leqslant 0$ (see (51)), but $h < 0$, and for this reason the fraction under the limit sign will itself be non-negative. But in that case the limit of this fraction (i. e. the derivative we are interested in) *cannot be negative.*

Thus, the value of the derivative $\frac{dy}{dx}$ for $x = a$ can be neither positive nor negative, therefore, it must be zero, and this proves our proposition.*)

In this proof the essential fact was that a is an *internal* point of the domain of definition of the function. Indeed, we attributed positive and negative values to h so that $a + h$ assumed values greater and less than a.

Suppose a is an end point. Then the domain of definition includes either values greater than a, or less than a, i. e. the above proof is inapplicable.

When dealing with the problem of the *minimum* (and not maximum) value of a function, the analysis is quite similar. As a result we

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*) It would certainly not be out of place to point out that if the limit exists, its value must be the same for $h > 0$ and $h < 0$, i. e. $\lim\limits_{h>0} = \lim\limits_{h<0}$ – *translator's note.*

shall prove that if a function assumes its minimum value at an internal point of its domain of definition, the function's derivative at this point will become zero. Combining both these cases – of maximum and minimum values – we obtain the Fermat's theorem, named after a French mathematician of the XVIIth century.

THEOREM *If a function assumes its maximum (or minimum) value at an internal point of its domain of definition, the function's derivative at this point becomes zero.*

This theorem serves as a basis for obtaining maximum and minimum values with the aid of differentiation. We must find the derivative of the function in question and the corresponding internal points of its domain of definition where the derivative becomes zero. One should look for the point where the function attains its maximum (or minimum) value among these points (where the derivative becomes zero) or among the end points of the domain of definition.

Example 11. A battery having an electromotive force E and an internal resistance r is connected to the terminals of a conductor (for instance, a heater). What should the conductor's resistance be so that it draws the maximum power from the battery?

Solution. Denote the conductor's resistance by R. In this case the total resistance of the circuit will be $R + r$, and, accordingly, the current passing through the circuit will be $i = \dfrac{E}{R + r}$

The power supplied by the battery to the conductor is expressed by the formula $W = i^2R$, i. e. $W = \dfrac{E^2R}{(R + r)^2}$ (53)

Hence, problem may be formulated as follows: for what R does function W as expressed by formula (53) attain its maximum value?

The domain of definition of the function W is the semi-axis $R \geqslant 0$ (we consider only the case of positive ohmic resistance). Find the derivative $\dfrac{dW}{dR}$:

$$\frac{dW}{dR} = \lim_{h \to 0} \frac{\dfrac{E^2(R + h)}{(R + h + r)^2} - \dfrac{E^2R}{(R + r)^2}}{h} =$$

$$= \lim_{h \to 0} \frac{E^2[(R + h)(R + r)^2 - R(R + h + r)^2]}{h(R + h + r)^2(R + r)^2} =$$

$$= \lim_{h \to 0} \frac{r^2 - R^2 - Rh}{(R + h + r)^2(R + r)^2} = \frac{r^2 - R^2}{(R + r)^4} = \frac{r - R}{(R + r)^3}$$

I or the derivative $\frac{dW}{dR}$, i. e. the fraction

$$\frac{r - R}{(R + r)^3}$$

to become zero, its numerator $r - R$ should become zero, i. e. it should be $R = r$.

Hence the power W can attain its maximum (or minimum) value either for $R = r$ or at the end point of its domain of definition $R = 0$. For $R = 0$ the power W is zero, too (this is the minimum value). Therefore the power can attain its maximum value only for $R = r$, i. e. when *the conductor's resistance is equal to the internal resistance of the battery.*

Shall the power really be at its maximum for $R = r$? Indeed, we have proved only that the power *may* attain its maximum value for $R = r$, but this in itself does not mean that it really will be so.

It is not difficult to make sure that when $R = r$, the power W is really at its maximum. Indeed, when $R = 0$, the power W is zero, too; when R is very great, the current i will be very small and hence the power will be small (for the voltage drop across the conductor's terminals does not exceed E). It is therefore clear that the power must attain its maximum value for some (not very great) value of R. But since the power *must* attain its maximum value (and it *may* do so *only* for $R = r$), therefore it stands to reason that we do, indeed, obtain the maximum power when $R = r$.

Example 12. The problem is how to build a steam boiler in the shape of a cylinder so that it will have the required volume V. It is desirable to keep the total surface of the boiler down to a minimum (in that case the minimum amount of metal will be required to build the boiler; besides the smaller is the surface of the boiler the less are the heat losses due to contact with the atmosphere). Find the best (optimum) dimensions of the boiler.

Solution. Denote the radius of the cylinder's base by R and its height by h. Then

$$V = \pi R^2 h$$

i. e.

$$h = \frac{V}{\pi R^2}$$

The cylindrical surface is $S = 2\pi R^2 + 2\pi Rh$, i. e.

$$S = 2\pi R^2 + \frac{2V}{R} \tag{54}$$

We must find out for what values of R the quantity S (which depends on R, i. e. is a function of radius R) assumes its minimum value. Find the derivative $\frac{dS}{dR}$:

$$\frac{dS}{dR} = \lim_{h \to 0} \frac{\left[2\pi (R + h)^2 + \frac{2V}{R + h}\right] - \left[2\pi R^2 + \frac{2V}{R}\right]}{h} =$$

$$= \lim_{h \to 0} \frac{2\pi (2Rh + h^2) - \frac{2Vh}{R(R + h)}}{h} =$$

$$= \lim_{h \to 0} \left[4\pi R + 2\pi h - \frac{2V}{R(R + h)}\right] = 4\pi R - \frac{2V}{R^2}$$

Equating the derivative $\frac{dS}{dR}$ to zero, we find: $R = \sqrt[3]{\frac{V}{2\pi}}$ and therefore

$$h = \frac{V}{\pi R^2} = \sqrt[3]{\frac{4V}{\pi}} = 2\sqrt[3]{\frac{V}{2\pi}} = 2R$$

In other words, *the height of the cylinder should be equal to its diameter.*

Do we really obtain the minimum value of the cylindrical surface for that value of R? This we can easily check. Indeed, for very great values of R the surface S is also very great (since the value of the first term in the expression for S is great – see (54)). For very small values of R the value of the surface S is also very great (this time it is the second term that accounts for this). Therefore for some intermediate value (not too great and not too small) of R the quantity S must assume its minimum value. But since the derivative $\frac{dS}{dR}$ becomes zero only for one value of R, this is the value of R to which the minimum surface of the cylinder corresponds.

We shall confine ourselves to these two examples. If instead, the reader will find many such examples in textbooks and problem

books. The reader may be advised to solve some of them provided he does not ignore the final stage, i. e. the proof that there is indeed a maximum or a minimum value at the point obtained. More perfect methods are described in higher mathematics courses which enable the decision to be made whether the function does indeed assume its minimum or maximum value at the point obtained. Besides, there are *rules for calculating derivatives.* The author worked on the assumption that the reader is not familiar with these rules, and for this reason the derivatives in the examples above have been obtained with the aid of direct calculations.

The Problem of Drawing a Tangent

Let L be a curve and M_0 a point on it. Let us discuss the problem of drawing a *tangent* to the curve L at point M_0. First let us discuss how the tangent is defined in mathematics. Choose point M also lying on the curve L and draw a straight line M_0M which we shall term a *secant*, for it intersects the curve L at least at two points M_0 and M. Should point M move along the curve L approaching point M_0 (Fig. 13 shows the sequence of positions

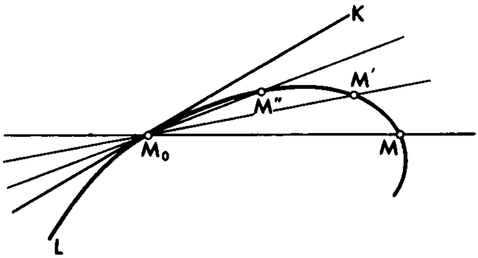

Fig 13

M, M', M'', ... of point M), the secant M_0M would turn around point M_0. If the secant M_0M in turning tends to coincide with some straight line M_0K, this limiting line M_0K is termed the *tangent* to the curve L at point M_0.

Suppose now that the curve L is drawn in a plane containing a coordinate system so that each point M of the curve L has its corresponding x- and y-coordinates. Denote the x-coordinate of point M_0 by a (Fig. 14), and the length of section N_0N by h. Then the x-coordinate of point M will be $a+h$. We denote the y-coordinate of point M_0 by y_a, and that of point M by

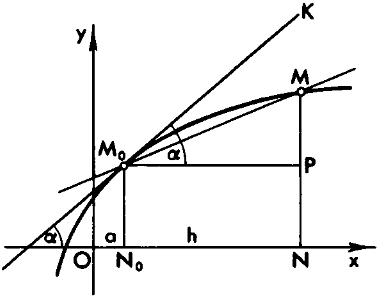

Fig 14

y_{a+h}. The length of section MP is

$$MP = MN - PN = MN - M_0N_0 = y_{a+h} - y_a$$

and we therefore have

$$\tan \angle PM_0M = \frac{MP}{M_0P} = \frac{MP}{N_0N} = \frac{y_{a+h} - y_a}{h} \tag{55}$$

Denote an angle PM_0K, i. e. the angle between the x-axis and the tangent, by α. Then as M approaches M_0, i. e. as a section $N_0N = h$ tends to zero, the angle PM_0M will approach α, and the tangent of the angle PM_0M $\tan\alpha$. Hence, we obtain from relation (55) in the limit (for $h \to 0$)

$$\tan\alpha = \lim_{h \to 0} \frac{y_{a+h} - y_a}{h} = \left.\frac{dy}{dx}\right|_{\text{for } x=a}$$

It follows that *the slope of the tangent line is equal to the value of the derivative of the y-coordinate with respect to the x-coordinate for $x = a$, where a is the point of tangency.*

Example 13. Consider a sine curve (Fig. 15), i. e. a curve

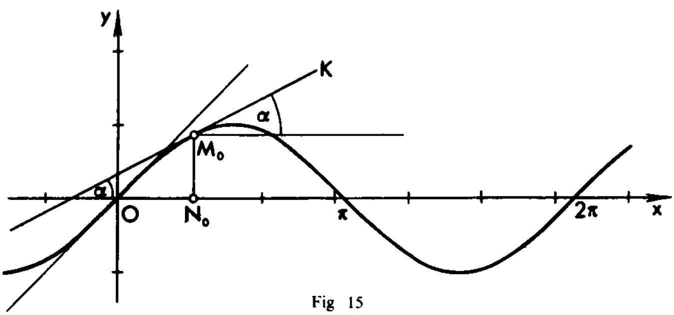

Fig 15

whose x- and y-coordinates are related by the equation

$$y = \sin x$$

How should a tangent line be drawn to this curve at some point M_0 whose x-coordinate is equal to a?

We know already how to find the slope of this tangent line

$$\tan\alpha = \left.\frac{dy}{dx}\right|_{\text{for } x=a} = \left.\frac{d}{dx}\sin x\right|_{\text{for } x=a} = \cos a$$

(see (36)). Hence, to draw a tangent line we must find $\cos a$ (which is quite easy since we know the length of section $M_0N_0 = y = \sin a$) and draw a straight line M_0K so that $\tan\alpha = \cos a$. For instance, when $a = 0$, we obtain: $\tan\alpha = \cos 0 = 1$, i. e. *the tangent line to a sine curve drawn through the origin of coordinates makes an angle* $\frac{\pi}{4}$ *with the x-axis.* For $a = \frac{\pi}{3}$ we obtain: $\tan\alpha = \cos\frac{\pi}{3} = \frac{1}{2}$, whence, using the trigonometric tables, we obtain that the angle α, which the tangent line makes with the x-axis, is equal approximately to $26°34'$.

Modelling

We have seen above that various physical phenomena which at first glance have nothing in common may be described by the same differential equations. That was the case with a body falling in a resistant medium, with the current being switched on in a circuit and with radioactive decay. This was the case with the three problems leading to harmonic oscillations. But when two phenomena are described by one equation, then the solution of this equation describes both phenomena. In other words, both phenomena take the same course (i. e. the values of the corresponding physical quantities change identically with time). Therefore we may, for example, study the electromagnetic oscillations in a circuit by observing the oscillations of a pendulum, and vice versa. This very simple remark proves to be very significant.

Suppose we have a design of an intricate machine before us and want to check whether the design was calculated correctly. It takes much time and money to build the machine, and still we must make sure beforehand that the design is the right one. Let us write out the equations of motion for the machine (these are usually differential equations). Should we solve these equations we would know how the machine is going to work and whether

it has been correctly designed. But it often proves easier to act as follows. Let us build another device (for instance, assemble an electric circuit) which would be described by the same differential equations as the machine to be designed. If we are able to do it, it will be sufficient to study the behaviour of the device we have built to learn how the machine we are designing is going to work. Thus, this device is a *model* of machine.

Such "modelling", too, uses the concept of the derivative as its basis, for to be able to build the right model one must know the differential equations which describe the operation of the machine.

Afterword

The concepts of the derivative and the differential equation are used extremely wide in mathematics, physics, astronomy, and engineering. The discipline that studies the properties and the applications of these concepts is "higher mathematics". Unfortunately, within the space of this booklet it would have been difficult to explain ideas underlying the definition of *integration* *) which, in a sense, is an operation reciprocal to differentiation and together with the latter forms the cornerstone of higher mathematics.

The author would like to draw the reader's attention to the point that the concepts of higher mathematics, including the concept of the derivative, are not abstract, but constitute a *mathematical reflection of processes taking place in nature* (for instance, of the speed of mechanical motion). These concepts developed in close contact with problems posed by everyday life – primarily in the fields of *mechanics* (the problem of calculating velocities) and *geometry* (the problem of drawing a tangent). F. Engels wrote: "Mathematics together with other sciences arose out of man's requirements." This statement fully applies to higher mathematics. The concept of the derivative created in conjunction with the study of the *motion* of bodies and the *variation* of quantities itself reflects this motion, for it treats a *variable quantity*. "The turning point in mathematics was Descartes' variable quantity. Thanks to it mathematics was invaded by *motion* and *dialectics* and it, too, created an *immediate need for differential and integral calculus* the rudiments of which were soon to be established and which was completed as an entity but not discovered by Newton and Leibnitz" (Engels).

*) See I. P. Natanson *Summirovanie beskonechno malykh velichin* (The Summation of Infinitesimals) published in this series (No. 12).

Thus the concepts of higher mathematics arose out of man's requirements, primarily in connection with the study of mechanical motion. But motion in nature is not only limited to its mechanical form. Time and again new phenomena reveal themselves to scientists, and mathematics has to describe newly discovered phenomena and forms of motion. Physical theories created in recent times (the theory of relativity, quantum physics, the theory of the nucleus) require new mathematical apparatus. Some mathematical disciplines were born during the last decades. *Mathematics is not an abstract science but a science closely connected with life which develops together with the growth of our knowledge of the physical world.*

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation, and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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